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Expressions of pattern sequences

By

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Abstract

In this paper, we investigate a basis of the module generated by special pattern sequences and study the expressions of pattern sequences using the basis. As an application, we give some linear relations between the pattern sequences in the balanced ternary.

§ 1. Pattern sequence

Let $q \geq 2$ and r be fixed integers with $r \in \{0, 1, \ldots, q-2\}$. Then any integer $n \geq 1$ is uniquely expressed as

$$n = \sum_{i=0}^{k-1} b_i q^i, \quad b_i \in \Sigma_{q,r}, \quad b_{k-1} > 0,$$

where $\Sigma_{q,r} := \{-r, 1-r, \ldots, 0, 1, \ldots, q-1-r\} \supset \{0, 1\}$. The string of $\langle q, r\rangle$-*digits* $(n)_{q,r} := b_{k-1} \cdots b_1 b_0$ is called the $\langle q, r\rangle$-*expansion* of n. The $\langle q, 0\rangle$-expansion is the ordinary q-ary expansion. These numeration systems are called $\langle q, r\rangle$-numeration systems and various aspects are discussed in, e.g., [1], [4], and [6].

A *word* or a *pattern* over $\Sigma_{q,r}$ is a finite string of elements in $\Sigma_{q,r}$. The set of all finite nonempty words is denoted by $\Sigma_{q,r}^*$. For $w = b_{l-1} \cdots b_1 b_0 \in \Sigma_{q,r}^*$ with $b_j \in \Sigma_{q,r}$, we define the length $|w| := l$. We write $w^k = ww \cdots w$ (k times), in particular w^0 denotes the empty word. For $w \in \Sigma_{q,r}^*$, we define $e_{q,r}(w; n)$ to be the number of (possibly overlapping) occurrences of w in the $\langle q, r\rangle$-expansion of n. Here if $w \neq 0^l$ for any $l \geq 1$, then in evaluating $e_{q,r}(w; n)$ we assume that the $\langle q, r\rangle$-expansion of n starts with arbitrary long strings of zeros. On the other hand, if $w = 0^l$ for some $l \geq 1$ we just count the number of occurrences of w in the $\langle q, r\rangle$-expansion of n. Define $e_{q,r}(w; 0) = 0$

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for any $w \in \Sigma_{q,r}^*$. The resulting sequence $\{e_{q,r}(w;n)\}_{n\geq 0}$ is sometimes called the *pattern sequence* for the pattern w in the $\langle q,r\rangle$-numeration system (cf. [1]).

Example 1.1. (i) $e_{2,0}(1;n)$ =the number of 1 in the base-2 expansion of n.
(ii) $\{e_{3,1}(1;n)\}_{n\geq 0} = \{0,1,1,1,2,1,1,2,1,1,2,2,2,3,1,\dots\}$.

Kirschenhofer [2] proved an asymptotic formula for the mean value of the average of $e_{q,0}(w;n)$ $(n=1,2,\dots,N)$. Similar results were obtained in any $\langle q,r\rangle$-numeration systems by Kirschenhofer and Prodinger [3]. Uchida [12] gave necessary and sufficient conditions for the generating functions of pattern sequences defined in a fixed q-ary numeration system to be algebraically dependent over $\mathbb{C}(z)$. This result was generalized by Shiokawa and the author [7] to any $\langle q,r\rangle$-numeration systems. Generating functions and their values defined by digital properties of integers have also been studied in [5], [9], [10], and [11].

For $w = b_{l-1}\cdots b_1 b_0 \in \Sigma_{q,r}^*$ with $b_i \in \Sigma_{q,r}$, we define $[w]_{q,r} := \sum_{i=0}^{l-1} b_i q^i$. Then the pattern sequence satisfies the following properties.

Lemma 1.2 ([8, Lemma 1]). *Let $w \in \Sigma_{q,r}^*$ with $|w| = l$. For any integer $n \geq 0$, we have*

$$e_{q,r}(w;n) = \sum_{b=-r}^{q-1-r} e_{q,r}(bw;n), \tag{1.1}$$

and if $w \neq 0^l$

$$e_{q,r}(w;n) = \sum_{b=-r}^{q-1-r} e_{q,r}(wb;n) + \delta(w;n), \tag{1.2}$$

where

$$\delta(w;n) = \begin{cases} 1 & \text{if } n \equiv [w]_{q,r} \pmod{q^l}, \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.3. In the 2-ary number system, we have by (1.2)

$$e_{2,0}(1;n) - e_{2,0}(10;n) - e_{2,0}(11;n) = \delta(n),$$

where $\{\delta(n)\}_{n\geq 0} := \{\dot{0},\dot{1}\}$ is a periodic sequence with a period whose length is two.

For any $w \in \Sigma_{q,r}^*$, the pattern sequence $\{e_{q,r}(w;n)\}_{n\geq 0}$ is not periodic. Indeed, $e_{q,r}(w,n_j) \to +\infty$ $(j \to +\infty)$ for positive integers n_j defined by $(n_j)_{q,r} = 1w^j$ $(j \geq 1)$. On the other hand, for some patterns $w_1,\dots,w_m \in \Sigma_{q,r}^*$ $(m \geq 2)$ the nontrivial linear combination over $\mathbb{Z}$ of the corresponding pattern sequences can be periodic (e.g. Example 1.3).

§ 2. Periodicity

In this section, we give necessary and sufficient conditions for a linear combination of pattern sequences to be periodic (cf. [8, Theorem 1]).

Theorem 2.1. *Let $m \geq 2$ and $w_1, \ldots, w_m \in \Sigma_{q,r}^*$ with $l = \max_{1\leq i\leq m} |w_i|$. Then the following three statements are equivalent*:

(i) *There exist $c_1, \ldots, c_m \in \mathbb{Z}$ not all zero such that $\{\sum_{i=1}^m c_i e_{q,r}(w_i; n)\}_{n\geq 0}$ is a periodic sequence.*

(ii) *There exist $c_1, \ldots, c_m \in \mathbb{Z}$ not all zero such that $\{\sum_{i=1}^m c_i e_{q,r}(w_i; n)\}_{n\geq 0}$ is a purely periodic sequence with a period whose length is q^{l-1}.*

(iii) *The rank of the matrix*

$$(e_{q,r}(w_i; n) - e_{q,r}(w_i; \overline{n}))_{q^{l-1}\leq n\leq q^l, 1\leq i\leq m} \tag{2.1}$$

is less than m, where $n \equiv \overline{n} \pmod{q^{l-1}}$ with $0 \leq \overline{n} < q^{l-1}$.

Remark 1. For any given patterns, the condition (iii) can be checked in finite steps. Furthermore, if the condition (iii) is satisfied, then we can find $c_1, \ldots, c_m \in \mathbb{Z}$ not all zero such that the sequence

$$\{c_1 e_{q,r}(w_1; n) + \cdots + c_m e_{q,r}(w_m; n)\}_{n\geq 0}$$

is periodic (cf. [8, Proof of Theorem 1]).

Example 2.2. We consider the so-called balanced ternary, namely, the $\langle 3, 1\rangle$-numeration system. For the patterns $w_1 := 01$, $w_2 := 10$, $w_3 := 1\overline{1}$, and $w_4 := \overline{1}1$ ($\overline{1} := -1 \in \Sigma_{3,1}$), the matrix (2.1) is given by

$$(e_{3,1}(w_i; n) - e_{3,1}(w_i; \overline{n}))_{3\leq n\leq 9, 1\leq i\leq 4} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

which is of the rank 3 (< 4). Hence, by Theorem 2.1, there exist $c_1, c_2, c_3, c_4 \in \mathbb{Z}$ not all zero such that

$$\{c_1 e_{3,1}(01; n) + c_2 e_{3,1}(10; n) + c_3 e_{3,1}(1\overline{1}; n) + c_4 e_{3,1}(\overline{1}1; n)\}_{n\geq 0}$$

is a purely periodic sequence with a period whose length is three. Here we can find such c_i by using the formulas (1.1) and (1.2). Indeed, we get by (1.1) and (1.2)

$$e_{3,1}(1;n) = e_{3,1}(\bar{1}1;n) + e_{3,1}(01;n) + e_{3,1}(11;n),$$

$$e_{3,1}(1;n) = e_{3,1}(1\bar{1};n) + e_{3,1}(10;n) + e_{3,1}(11;n) + \delta(n),$$

with $\{\delta(n)\}_{n\geq 0} = \{\dot{0}, 1, \dot{0}\}$, respectively, so that

$$\{e_{3,1}(01;n) - e_{3,1}(10;n) - e_{3,1}(1\bar{1};n) + e_{3,1}(\bar{1}1;n)\}_{n\geq 0} = \{\dot{0}, 1, \dot{0}\}.$$

Example 2.3. Any nontrivial linear combination over $\mathbb{Z}$ of three pattern sequences in $\{e_{3,1}(w_i;n)\}_{n\geq 0}$ $(i = 1,2,3,4)$ can not be periodic. In particular, any three pattern sequences in $\{e_{3,1}(w_i;n)\}_{n\geq 0}$ $(i = 1,2,3,4)$ are linearly independent over $\mathbb{Z}$ as a sequence.

§ 3. The module M_l

In what follows, let $P_k(\mathbb{Z})$ be the set of all purely periodic sequences of integers $\{\alpha(n)\}_{n\geq 0}$ with $\alpha(0) = 0$ and a period whose length is q^k. Note that $P_{k-1}(\mathbb{Z}) \subset P_k(\mathbb{Z})$ $(k \geq 1)$. Denote by M_l the module generated by the pattern sequences $\{e_{q,r}(w;n)\}_{n\geq 0}$ for $w \in \Sigma^*_{q,r}$ with $|w| \leq l$.

Proposition 3.1. *For any $l \geq 1$, the module M_l contains $P_{l-1}(\mathbb{Z})$ as a submodule and there is no other periodic sequence in M_l.*

Proof of Proposition 3.1. The assertion is trivial for $l = 1$. Let $l \geq 2$. For any $w \in \Sigma^*_{q,r}$ with $|w| = l-1$ and $w \neq 0^{l-1}$, we have by (1.1) and (1.2)

$$\sum_{a=-r}^{q-1-r} e_{q,r}(aw;n) - \sum_{b=-r}^{q-1-r} e_{q,r}(wb;n) = \delta(w;n), \tag{3.1}$$

where $|aw| = |wb| = l$ and $\delta(w;n) \in P_{l-1}(\mathbb{Z})$. Let $\{\lambda_i(n)\}_{n\geq 0}$ $(i = 1,\dots,q^{l-1}-1)$ denotes the purely periodic sequence with a period whose length is q^{l-1} such that $\lambda_i(i) = 1$ and $\lambda_i(j) = 0$ $(j \neq i)$ for $j = 0,1,\dots q^{l-1}-1$. Clearly, these sequences generate $P_{l-1}(\mathbb{Z})$. By definition, $\{\delta(w;n)\}_{n\geq 0} = \{\lambda_i(n)\}_{n\geq 0}$ if $[w]_{q,r} \equiv i \pmod{q^{l-1}}$. Putting $S = \{w \in \Sigma^*_{q,r} \mid |w| = l-1, w \neq 0^{l-1}\}$, we see that $[w]_{q,r} \not\equiv [w']_{q,r} \pmod{q^{l-1}}$ for $w, w' \in S$ with $w \neq w'$ and $\sharp S = q^{l-1}-1$. Hence we deduce from (3.1) $\{\lambda_i(n)\}_{n\geq 0} \in M_l$ $(i = 1,2,\dots,q^{l-1}-1)$, and therefore $P_{l-1}(\mathbb{Z}) \subset M_l$. Furthermore, if $\{\sum_{i=1}^m c_i e_{q,r}(w_i;n)\}_{n\geq 0} \in M_l$ is a periodic sequence of integers, then by Theorem 2.1 (i)$\Leftrightarrow$(ii) it must be purely periodic with a period whose length is q^{k-1}, where $k = \max_{1\leq i\leq m} |w_i| \leq l$. Noting that $P_{k-1}(\mathbb{Z}) \subset P_{l-1}(\mathbb{Z})$, we have the conclusion. $\square$

Example 3.2. Let $q = 3$, $r = 1$, and $l = 3$. Putting $\{e_{3,1}(w;n)\}_{n\geq 0} := (w;n)$ for brevity, we have

$$\begin{aligned}
&(\overline{1}01;n) + (001;n) + (101;n) - (01\overline{1};n) - (010;n) - (011;n) = \{\dot{0},1,0,0,0,0,0,0,\dot{0}\},\\
&(\overline{1}1\overline{1};n) + (01\overline{1};n) + (11\overline{1};n) - (1\overline{11};n) - (1\overline{1}0;n) - (1\overline{1}1;n) = \{\dot{0},0,1,0,0,0,0,0,\dot{0}\},\\
&(\overline{1}10;n) + (010;n) + (110;n) - (10\overline{1};n) - (100;n) - (101;n) = \{\dot{0},0,0,1,0,0,0,0,\dot{0}\},\\
&(\overline{1}11;n) + (011;n) \qquad\qquad - (11\overline{1};n) - (110;n) \qquad\qquad = \{\dot{0},0,0,0,1,0,0,0,\dot{0}\},\\
&\qquad (0\overline{11};n) + (1\overline{11};n) \qquad\qquad - (\overline{11}0;n) - (\overline{11}1;n) = \{\dot{0},0,0,0,0,1,0,0,\dot{0}\},\\
&(\overline{11}0;n) + (0\overline{1}0;n) + (1\overline{1}0;n) - (\overline{1}0\overline{1};n) - (\overline{1}00;n) - (\overline{1}01;n) = \{\dot{0},0,0,0,0,0,1,0,\dot{0}\},\\
&(\overline{11}1;n) + (0\overline{1}1;n) + (1\overline{1}1;n) - (\overline{1}1\overline{1};n) - (\overline{1}10;n) - (\overline{1}11;n) = \{\dot{0},0,0,0,0,0,0,1,\dot{0}\},\\
&(\overline{1}0\overline{1};n) + (00\overline{1};n) + (10\overline{1};n) - (0\overline{11};n) - (0\overline{1}0;n) - (0\overline{1}1;n) = \{\dot{0},0,0,0,0,0,0,0,\dot{1}\}.
\end{aligned}$$

§ 4. Expressions of pattern sequences

In this section, we introduce the expression of pattern sequence by using some basis. First, we give a basis of the module generated by pattern sequences for words of length not exceeding l and study the expressions of pattern sequences using the basis. Similar results are obtained for the module generated by all pattern sequences. The proofs of the following Theorems 4.1–4.4 are given in [8].

For any integer $l \geq 1$, we put

$$V_l := \{w = b_{l-1}\cdots b_1 b_0 \in \Sigma_{q,r}^* \mid b_i \in \Sigma_{q,r}, b_{l-1} \neq 0\} \cup \{0^l\},$$

where $\sharp V_l = (q-1)q^{l-1} + 1$.

Theorem 4.1. *Let $l \geq 1$ be a fixed integer. The module M_l is generated by the pattern sequences $\{e_{q,r}(w;n)\}_{n\geq 0}$ for $w \in V_l$ mod $P_{l-1}(\mathbb{Z})$. More precisely, for any $w \in \Sigma_{q,r}^*$ with $w \leq l$, there exist distinct patterns $w_1, \ldots, w_m \in V_l$ and nonzero integers $c_1, \ldots, c_m$ such that*

$$\{e_{q,r}(w;n)\}_{n\geq 0} \equiv \left\{\sum_{i=1}^{m} c_i e_{q,r}(w_i;n)\right\}_{n\geq 0} \mod P_{l-1}(\mathbb{Z}), \tag{4.1}$$

where $c_1, \ldots, c_m \in \{1, -1\}$ if $w \neq 0^k$ for any $k \leq l-2$. Furthermore, the pattern sequences $\{e_{q,r}(w;n)\}_{n\geq 0}$ ($w \in V_l$) are linearly independent over $\mathbb{Z}$ mod $P_{l-1}(\mathbb{Z})$.

Theorem 4.2. *Let $w \in \Sigma_{q,r}^*$ with $|w| = k \leq l$. Then the number $m = m(w)$ of words in V_l necessary for the expression* (4.1) *is bounded by, if $q = 2$,*

$$m = m(w) \leq \begin{cases} 2^{l-k} & (k = 1, 2), \\ 2^{l-2} + 2^{l-k} & (k = 3, \ldots, l), \end{cases} \tag{4.2}$$

and, if $q \geq 3$,

$$m = m(w) \leq \begin{cases} q^{l-1} & (k=1), \\ 2q^{l-1} - 3q^{l-2} & (k=2), \\ 2\left(1 - \frac{1}{q-1}\right) q^{l-1} - \left(1 - \frac{2}{q-1}\right) q^{l-k} & (k = 3, \ldots, l), \end{cases} \tag{4.3}$$

where the equalities in (4.2) $(k = 1, 2)$, (4.2) $(3 \leq k \leq l)$, *and* (4.3) $(1 \leq k \leq l)$ *hold for* $w = 0^{k-1}1$, $w = 0^{k-2}10$, *and* $w = 0^{k-1}1$, *respectively.*

For $l \geq 1$, we define

$$U_l := \{w = b_{l-1} \cdots b_1 b_0 \in \Sigma_{q,r}^* \mid b_i \in \Sigma_{q,r},\ b_0 \neq 0, b_{l-1} \neq 0\} \subset V_l.$$

We denote by M the module generated by all pattern sequences $\{e_{q,r}(w;n)\}_{n\geq 0}$ for $w \in \Sigma_{q,r}^*$, so that $M = \cup_{l=1}^{\infty} M_l$. By Proposition 3.1, the module M contains $\cup_{l=1}^{\infty} P_{l-1}(\mathbb{Z})$ as a submodule and there is no other periodic sequence in M.

Theorem 4.3. *The module M is generated by the pattern sequences $\{e_{q,r}(w;n)\}_{n\geq 0}$ for $w \in \cup_{j=1}^{\infty} U_j \cup \{0\}$ mod $\cup_{l=1}^{\infty} P_{l-1}(\mathbb{Z})$. More precisely, for any $w \in \Sigma_{q,r}^*$ with $|w| = l$, there exist distinct patterns $w_1, \ldots, w_m \in \cup_{j=1}^{l} U_j \cup \{0\}$ and nonzero integers $c_1, \ldots, c_m$ such that*

$$\{e_{q,r}(w;n)\}_{n\geq 0} \equiv \left\{ \sum_{i=1}^{m} c_i e_{q,r}(w_i;n) \right\}_{n\geq 0} \mod P_{l-1}(\mathbb{Z}). \tag{4.4}$$

Furthermore, for distinct $w_1, \ldots, w_s \in \cup_{j=1}^{\infty} U_j \cup \{0\}$, the pattern sequences $\{e_{q,r}(w_1;n)\}_{n\geq 0}, \ldots, \{e_{q,r}(w_s;n)\}_{n\geq 0}$ are linearly independent over $\mathbb{Z}$ mod $P_{l-1}(\mathbb{Z})$, where $l = \max\{|w_1|, \ldots, |w_s|\}$.

For a real number t, $\lfloor t \rfloor$ denotes the greatest integer not exceeding t.

Theorem 4.4. *Let $w \in \Sigma_{q,r}^*$ with $|w| = l \geq 1$. Then the number $m = m(w)$ of words in $\cup_{j=1}^{l} U_j \cup \{0\}$ necessary for the expression* (4.4) *is bounded by*

$$m = m(w) \leq \begin{cases} 1 & (l=1), \\ 1 + (q-1) + (l-1)(q-1)^2 & (l = 2,3,4), \\ 1 + (q-1)(l-1) + \left\lfloor \frac{l-1}{2} \right\rfloor \left((q-1)^2 \left\lfloor \frac{l}{2} \right\rfloor - 1 \right) & (l \geq 5), \end{cases}$$

where the equalities hold for $w = 0^l$ $(l = 1,2,3,4)$ *and* $w = 0^j 1 0^k$ *with* $j = \lfloor (l-1)/2 \rfloor$ *and* $k = \lfloor l/2 \rfloor$ $(l \geq 5)$.

Example 4.5. By Theorem 4.1, $\{e_{3,1}(01;n)\}_{n\geq 0}$ can be written as a sum of pattern sequences for words in V_j and a purely periodic sequence with a period whose length is 3^{j-1}, where $j \geq 2$. If $j = 2, 3$ we have

$$\begin{aligned} e_{3,1}(01;n) &= e_{3,1}(1\bar{1};n) + e_{3,1}(10;n) - e_{3,1}(\bar{1}1;n) + \alpha(n), \\ &= e_{3,1}(1\bar{1}\bar{1};n) + e_{3,1}(1\bar{1}0;n) + e_{3,1}(1\bar{1}1;n) + e_{3,1}(10\bar{1};n) + e_{3,1}(100;n) \\ &\quad + e_{3,1}(101;n) - e_{3,1}(\bar{1}1\bar{1};n) - e_{3,1}(\bar{1}10;n) - e_{3,1}(\bar{1}11;n) + \beta(n), \end{aligned}$$

where $\{\alpha(n)\}_{n\geq 0} := \{\dot{0}, 1, \dot{0}\}$ and $\{\beta(n)\}_{n\geq 0} := \{\dot{0}, 1, 1, 1, 1, 0, 0, 0, \dot{0}\}$. This example implies also that pattern sequences for distinct words in $V_i \cup V_j$ $(i < j)$ can be linearly dependent over $\mathbb{Z}$ modulo $P_{j-1}(\mathbb{Z})$. On the other hand, Theorem 4.3 gives the expression

$$e_{3,1}(01;n) = e_{3,1}(1;n) - e_{3,1}(\bar{1}1;n) - e_{3,1}(11;n).$$

The expression (4.4) in Theorem 4.3 admits c_j with $|c_j| \geq 2$ even in the case $w \neq 0^l$, contrary to that of (4.1) in Theorem 4.1. For example, for $w = 010 \neq 0^3$ we have

$$\begin{aligned} e_{3,1}(010;n) &= e_{3,1}(1;n) - e_{3,1}(\bar{1}1;n) - e_{3,1}(1\bar{1};n) - 2e_{3,1}(11;n) + e_{3,1}(\bar{1}1\bar{1};n) \\ &\quad + e_{3,1}(\bar{1}11;n) + e_{3,1}(11\bar{1};n) + e_{3,1}(111;n) - \gamma(n), \end{aligned}$$

where $\gamma(n) = 1$ if $n \equiv 1 \pmod 9$, $= 0$ otherwise.

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